Recent advances in fluid boundary layer theory

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Journées de Physique Statistique
January 30th&31st, 2020 - Paris

European Research Council
Established by the European Commission

Outline

Plan

Fluids with small viscosity

Goal: understand the behavior of 2d fluids with small viscosity in a domain $\Omega \subset \mathbf{R}^2$.

$$\begin{aligned} \partial_t \mathbf{u}^\nu + (\mathbf{u}^\nu \cdot \nabla)\mathbf{u}^\nu + \nabla p^\nu - \nu \Delta \mathbf{u}^\nu &= 0 \text{ in } \Omega, \\ \operatorname{div} \mathbf{u}^\nu &= 0 \text{ in } \Omega, \\ \mathbf{u}^\nu_{|\partial\Omega} = 0, \quad \mathbf{u}^\nu_{|t=0} &= \mathbf{u}^\nu_{ini}. \end{aligned} \tag{1}$$

$\rightarrow$ Singular perturbation problem.
Formally, if $\mathbf{u}^\nu \rightarrow \mathbf{u}^E$, and if $\Delta \mathbf{u}^\nu$ remains bounded, then $\mathbf{u}^E$ is a solution of the Euler system

$$\begin{aligned} \partial_t \mathbf{u}^E + (\mathbf{u}^E \cdot \nabla)\mathbf{u}^E + \nabla p^E &= 0 \text{ in } \Omega, \\ \operatorname{div} \mathbf{u}^E &= 0 \text{ in } \Omega. \end{aligned} \tag{2}$$

But what about boundary conditions?

Boundary conditions

- **Navier-Stokes:** parabolic system.

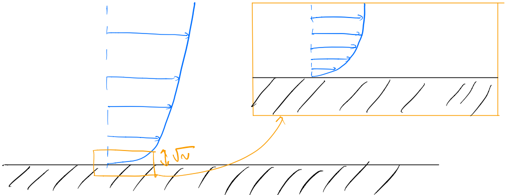
$\rightarrow$ Dirichlet boundary conditions can be enforced: $\mathbf{u}^\nu_{|\partial\Omega} = 0$.

- **Euler:** $\sim$ hyperbolic system, with a divergence-free condition $\operatorname{div} \mathbf{u}^E = 0$.

$\rightarrow$ Condition on the normal component only (non-penetration condition): $\mathbf{u}^E \cdot n_{|\partial\Omega} = 0$.

Consequence:

- Loss of the tangential boundary condition as $\nu \to 0$;
- Formation of a boundary layer in the vicinity of $\partial\Omega$ to correct the mismatch between $0 (= \mathbf{u}^\nu \cdot \tau_{|\partial\Omega})$ and $\mathbf{u}^E \cdot \tau_{|\partial\Omega}$.



The half-space case: Prandtl's Ansatz

[Prandtl, 1904] in the limit $\nu \ll 1$, if $\Omega = \mathbf{R}^2_+$,

$$\mathbf{u}^\nu(t,x,y) \simeq \begin{cases} \mathbf{u}^E(t,x,y) \text{ for } y \gg \sqrt{\nu} \text{ (sol. of 2d Euler)}, \\ \left(u^P\left(t,x,\frac{y}{\sqrt{\nu}}\right), \sqrt{\nu}v^P\left(t,x,\frac{y}{\sqrt{\nu}}\right)\right) \text{ for } y \lesssim \sqrt{\nu}. \end{cases}$$

The velocity field (u^P, v^P) satisfies the Prandtl system

$$\partial_t u^P + u^P \partial_x u^P + v^P \partial_Y u^P - \partial_{YY} u^P = -\frac{\partial p^E}{\partial x}(t,x,0)$$

$$\partial_x u^P + \partial_Y v^P = 0,$$

$$\mathbf{u}^P_{|Y=0} = 0, \quad \lim_{Y\to\infty} u^P(t,x,Y) = u_\infty(t,x) := u^E(t,x,0),$$

$$u^P_{|t=0} = u^P_{ini}.$$

The Prandtl equation: general remarks

$$
\left\{
\begin{aligned}
\partial_t u^P + u^P \partial_x u^P + v^P \partial_Y u^P - \partial_{YY} u^P &= -\frac{\partial p^E}{\partial x}(t,x,0) \\
\partial_x u^P + \partial_Y v^P &= 0, \\
\mathbf{u}^P_{|Y=0} = 0, \ \lim_{Y\to\infty} u^P(t,x,Y) = u_\infty(t,x) &:= u^E(t,x,0), \\
u^P_{|t=0} &= u^P_{ini}.
\end{aligned}
\right. \qquad \text{(P)}
$$

Comments:

- Nonlocal, scalar equation: write $v^P = -\int_0^Y u^P_x$;
- Pressure is given by Euler flow= data;
- Main source of trouble: nonlocal transport term $v^P \partial_Y u^P$ (loss of one derivative).

Questions around the Prandtl system

1. Is the Prandtl system well-posed? (i.e. does there exist a unique solution?) In which function spaces? Under which conditions on the initial data?
2. When the Prandtl system is well-posed, can we justify the Prandtl Ansatz? i.e. can we prove that

$$\|\mathbf{u}^\nu - \mathbf{u}^\nu_{\text{app}}\| \to 0 \text{ as } \nu \to 0$$

   in some suitable function space, where the function $\mathbf{u}^\nu_{\text{app}}$ is such that

$$\mathbf{u}^\nu_{\text{app}}(t,x,y) \simeq \begin{cases} \mathbf{u}^E(t,x,y) \text{ for } y \gg \sqrt{\nu} \\ \left(u^P\left(t,x,\frac{y}{\sqrt{\nu}}\right), \sqrt{\nu}v^P\left(t,x,\frac{y}{\sqrt{\nu}}\right)\right) \text{ for } y \lesssim \sqrt{\nu}. \end{cases}$$

Function spaces

- **L^2 space:** $\|u\|_{L^2(\Omega)} = \left(\int_\Omega |u|^2\right)^{1/2}$.
- **Sobolev spaces H^s, $s \in \mathbf{N}$:** $\|u\|_{H^s} = \sum_{|k|\le s} \|\nabla^k u\|_{L^2}$.

($\sim$ Polynomial decay of Fourier modes)

- **Space of analytic functions:** $\exists C > 0$, s.t. for all $k \in \mathbf{N}^d$,

$$\sup_{x\in\Omega} |\nabla^k u(x)| \le C^{|k|+1}|k|!.$$

($\sim$ Exponential decay of Fourier modes)

- **Gevrey spaces G^τ, $\tau > 0$:** $\exists C > 0$, s.t. for all $k \in \mathbf{N}^d$,

$$\sup_{x\in\Omega} |\nabla^k u(x)| \le C^{|k|+1}(|k|!)^\tau.$$

($\sim$ Fourier modes decay like $\exp(-c|k|^{1/\tau})$)

If $\tau > 1$, G^τ contains non trivial functions with compact support.

Plan

Well-posedness under positivity assumptions

Stationary Prandtl system:

$$
\begin{aligned}
u\partial_x u + v\partial_Y u - \partial_{YY} u = -\frac{\partial p^E}{\partial x}(x,0)\\
\partial_x u + \partial_Y v = 0, \quad u_{|x=0} = u_0\\
u_{|Y=0} = 0, \quad v_{|Y=0} = 0, \quad \lim_{Y\to\infty} u(x,Y) = u_\infty(x).
\end{aligned}
\qquad \text{(SP)}
$$

$\sim$ Non-local, "transport-diffusion" equation .

Theorem [Oleinik, 1962]: *Let $u_0 \in \mathcal{C}_b^{2,\alpha}(\mathbf{R}_+)$, $\alpha > 0$. Assume that $u_0(Y) > 0$ for $Y > 0$, $u_0'(0) > 0$, $u_\infty > 0$ + compatibility condition.*
Then there exists $x^* > 0$ such that (SP) has a unique $\mathcal{C}^2$ solution in $\{(x,Y) \in \mathbf{R}^2,\ 0 \le x < x^*,\ 0 \le Y\}$. If $\frac{\partial p^E(x,0)}{\partial x} \le 0$, then $x^* = +\infty$.

Well-posedness under positivity assumptions

Stationary Prandtl system:

$$u\partial_x u + v\partial_Y u - \partial_{YY} u = -\frac{\partial p^E}{\partial x}(x,0)$$
$$\partial_x u + \partial_Y v = 0, \quad u_{|x=0} = u_0 \qquad \text{(SP)}$$
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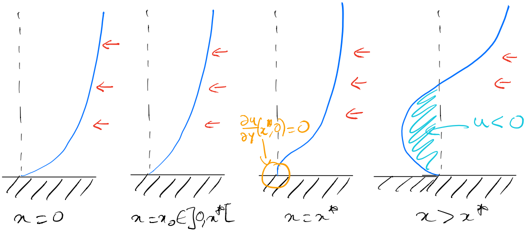
Comments on Oleinik's theorem

- The solution lives as long as there is no recirculation, i.e. as long as u remains positive.
- Maximal existence interval $(0, x^*)$: if $x^* < +\infty$, then
  (i) either $\partial_Y u(x^*, 0) = 0$
  (ii) or $\exists Y^* > 0,\ u(x^*, Y^*) = 0$.
- Monotony (in Y) is preserved by the equation. If u_0 is monotone, scenario (ii) cannot happen.

Illustration(s) of the "separation" phenomenon



Separation point: $\frac{\partial u}{\partial Y}_{|x=x^*,Y=0} = 0.$

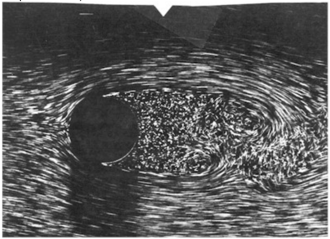


Figure: Cross-section of a flow past a cylinder (source: ONERA, France)

Goldstein singularity

- Formal computations of a solution by [Goldstein '48, Stewartson '58] (asymptotic expansion in well-chosen self-similar variables; see also [Landau, '59]).
  Prediction: there exists a solution such that $\partial_Y u_{|Y=0}(x) \sim \sqrt{x^* - x}$ as $x \to x^*$.
- [D., Masmoudi, '18]: rigorous justification of the Goldstein singularity. Computation of an approximate solution, using modulation of variables techniques.
  Open problem: is $\sqrt{x^* - x}$ the "stable" separation rate?
- **Why "singularity"?**
  Since $v = -\int_0^Y u_x$, v becomes infinite as $x \to x^*$: separation.
- In this case, "generically", recirculation causes separation.

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Open problems for the stationary case

- Remove Goldstein singularity by adding corrector terms in the equation, coming from the coupling with the outer flow (triple deck system?);
- Construct solutions with recirculation (work in progress... Idea: construct solutions in the vicinity of explicit recirculating flows).

Justification of the Prandtl Ansatz

Overall idea: far from the separation point, as long as there is no re-circulation, the Prandtl Ansatz can be justified.

- [Guo& Nguyen, '17]: Navier-Stokes system above a moving plate (non-zero boundary condition), later extended by [Iyer];
- [Gérard-Varet& Maekawa, '18]: main order term in Prandtl is a shear flow;
- [Guo& Iyer, '18]: main order term in Prandtl is the Blasius boundary layer (self-similar solution).

Plan

A reminder...

Time-dependent Prandtl equation (P):

$$
\begin{aligned}
\partial_t u + u\partial_x u + v\partial_Y u - \partial_{YY} u = -\frac{\partial p^E}{\partial x}(t,x,0)\\
\partial_x u + \partial_Y v = 0,\\
\mathbf{u}_{|Y=0} = 0,\ \lim_{Y\to\infty} u(t,x,Y) = u_\infty(t,x) := u^E(t,x,0),\\
u_{|t=0} = u_{ini}.
\end{aligned}
$$

$\sim$ (Degenerate) heat equation $\partial_t u - \partial_{YY} u$
$+$ local transport term $u\partial_x u$
$+$ non-local transport term with loss of one derivative

$$v\partial_Y u = -\int_0^Y u_x.$$

Mathematical results: well-posedness in high regularity spaces/monotonic contexts...

WP in high regularity spaces:

- Local well-posedness starting from data that are analytic in x: [Sammartino&Caflisch, '98; Lombardo, Cannone &Sammartino; Kukavica&Vicol; Kukavica, Masmoudi, Vicol&Wong];
- Extensions (e.g. Gevrey spaces): [Kukavica& Vicol, '13; Gérard-Varet& Masmoudi, '14; Maekawa, '14]

WP for monotone solutions: [Oleinik; Masmoudi&Wong; Alexandre, Wang, Xu&Yang...]

... and instabilities in Sobolev spaces

- **Instabilities** develop in short time **in Sobolev spaces** [Grenier; Gérard-Varet&Dormy; Grenier&Nguyen...]
  Proof relies on computation of an approximate solution whose k'th Fourier mode grows like $\exp(\sqrt{|k|}t)$.
- Starting from real analytic initial data, some solutions display **singularities in finite time** (van Dommelen-Shen singularity).
  [E& Engquist, Kukavica, Vicol&Wang]: virial type argument (blow-up of some Sobolev norm in finite time).
  Very recently, quantitative description of this singularity [Collot, Ghoul, Ibrahim&Masmoudi].
- **The Prandtl Ansatz is invalid in Sobolev spaces**, starting from an initial data for (NS) of the form $(U_s(y/\sqrt{\nu}), 0)$ [Grenier '00; Grenier, Guo& Nguyen, '16; Grenier, & Nguyen, '18].

Interactive boundary layer models

Intuition: [Catherall& Mangler; Le Balleur; Carter; Veldman...]
When a singularity is formed in the Prandtl system and the expansion ceases to be valid, the coupling with the interior flow must be considered at a higher order in ν, with potential stabilizing effects.

Cornerstone: notion of blowing velocity/displacement thickness:

$$v^P(t,x,Y) = -\int_0^Y u_x^P = -Y\partial_x u_\infty - \underbrace{\partial_x \int_0^Y (u^P - u_\infty)}_{=\text{"blowing velocity"}}.$$

Interactive boundary layer model: couple the Euler and the boundary layer systems by prescribing

$$v^E(t,x,0) = \sqrt{\nu}\partial_x \int_0^\infty (u_\infty - u^P(t,x,Y))\, dY.$$

Bad news: even worse than Prandtl! [D., Dietert, Gérard-Varet, Marbach, '17]

Summary

• **Stationary case:** the only mathematical setting in which solutions are known up to now is the case of positive solutions. For such a setting, we have a good understanding of singularities close to the separation point, and we are able to justify the Ansatz far from the separation.

• **Time-dependent case:** WP in high regularity settings and for monotone data.
In the non-monotone case, strong instabilities develop in Sobolev spaces; the boundary layer Ansatz fails.

Conclusion

• Small scale structures (both in x AND y) appear close to the wall in general (cf. instabilities): vortices.

• The boundary layer Ansatz should be replaced by something else, accounting for small scale vortices. But... what ? Statistical description?

THANK YOU FOR YOUR ATTENTION !